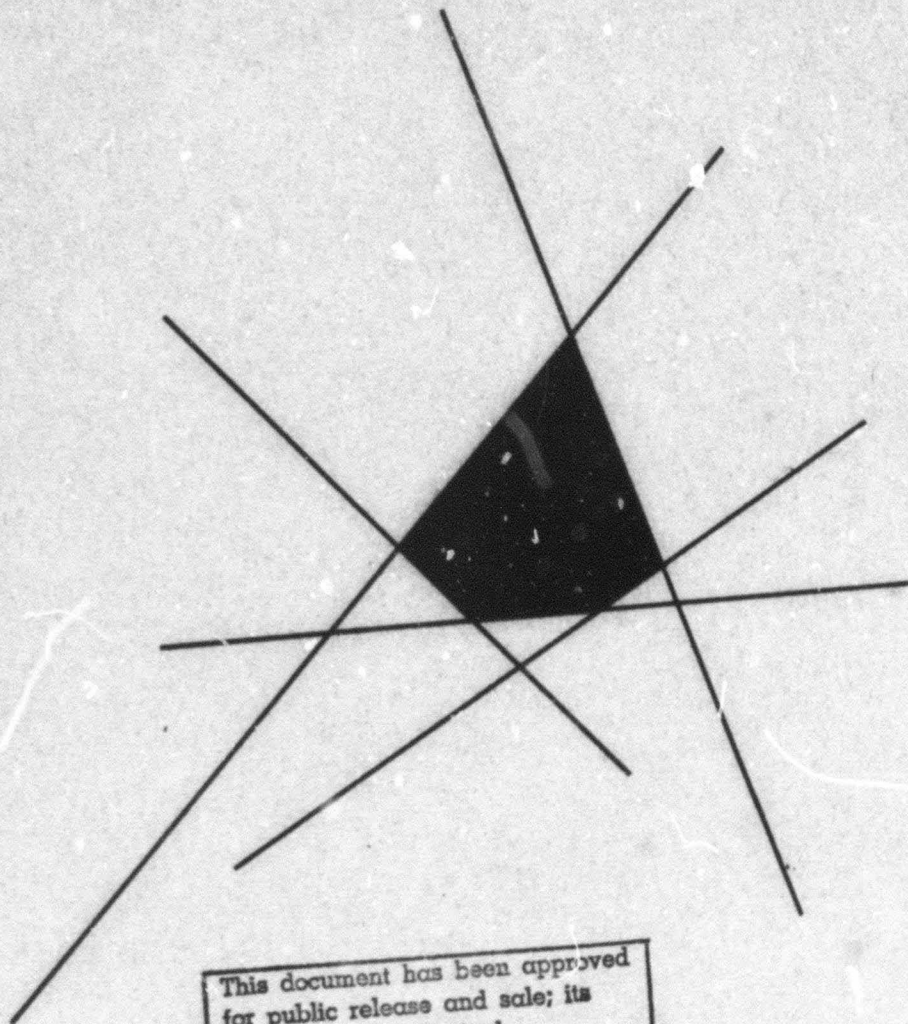


# APPROXIMATIONS TO SYSTEM RELIABILITY USING A MODULAR DECOMPOSITION

by

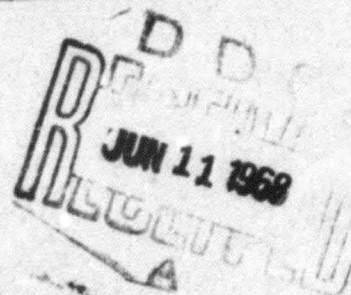
LAWRENCE D. BODIN

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APPROXIMATIONS TO SYSTEM RELIABILITY USING A  
MODULAR DECOMPOSITION

by

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Operations Research Center  
University of California, Berkeley

July 1967

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# ERRATA

## APPROXIMATIONS TO SYSTEM RELIABILITY USING A MODULAR DECOMPOSITION

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Page 5, line 7

Replace  $h(p) = h_{\phi}(p) \geq h^*(p)$  by  $h(p) = h_{\phi}(p) \geq h_{\phi}^*(p)$

Page 7, line 9

Replace by (6.21) of by by (6.1.2) of

Page 8, line 6

Replace  $A_1 = (C_{21-1}, C_{21})$  by  $A_1 = (c_{21-1}, c_{21})$

Page 10, line 10

Replace  $h_{\psi}^*[h_1^*(p), h_2^*(p)]$  by  $h_{\psi}^*[h_1^*(p), h_2^*(p)]$

Page 11, line 10

Replace  $= \psi[\sigma_1(w_{11}(x), \dots, w_{1s_1}(x)), \dots, \sigma_t(w_{t1}(x), \dots, w_{ts_t}(x))]$

by  $= \psi[\sigma_1(w_{11}(x), \dots, w_{1s_1}(x)), \dots, \sigma_t(w_{t1}(x), \dots, w_{ts_t}(x))]$

Page 12, line 15

Replace functions  $w_{ij}(X)$  by functions  $w_{ij}(x)$

Page 12, line 17

Replace  $\phi(x) = \psi_k[\chi_{1k}(x), \dots, \chi_{r_k k}]$ ,  $k=0, \dots, k^*$

by  $\phi(x) = \psi_k[\chi_{1k}(x), \dots, \chi_{r_k k}(x)]$ ,  $k=0, \dots, k^*$

#### ACKNOWLEDGMENT

I wish to thank Dr. James D. Esary of the Boeing Scientific Research Laboratory for his many helpful suggestions in developing and proving the results in this paper, and Professor Richard E. Barlow of the University of California, Berkeley, for his motivation and assistance.

#### ABSTRACT

Esary and Proschan show that a lower bound to system reliability can be found by enumerating all min cut sets in the coherent structure, connecting the components in each min cut set in parallel and joining each of these parallel subsystems in series where replicated components are replaced by identical yet independently operating components. A module of a coherent structure is a subset of the basic components of the system which can be treated as a component of the system due to their substructure topology.

In this paper, it is shown that a lower bound estimate of system reliability can be derived by decomposing the coherent structure about its modules and applying the Esary-Proschan lower bound procedure to each module and then to the resultant coherent structure where each module has been replaced by a single component whose reliability is the Esary-Proschan lower bound to that module.

This estimate of system reliability is sharper than the estimate of system reliability obtained by utilizing the Esary-Proschan procedure on the total system directly. Furthermore, this estimate is computationally more efficient than applying the Esary-Proschan procedure to the total system directly since the min cut sets need only be enumerated for each module. Applications of this result are given and analogous results for an upper bound to system reliability are stated.

# APPROXIMATIONS TO SYSTEM RELIABILITY USING A MODULAR DECOMPOSITION

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## 1.0 INTRODUCTION

A coherent structure  $(C, \phi)$  is made up of a set of components  $C = \{c_i\}_{i=1}^n$  which exist in one of two states--working or failed. Let  $x_i$  be a binary variable which designates the states of component  $i$ ;  $x_i = 1$  if the component works and 0 if failed. Similarly, define the structure function  $\phi(x) = \phi(x_1, \dots, x_n)^{\dagger}$  to be 1 if the system is working and 0 otherwise. For the structure to be coherent, the following two conditions must be satisfied:

- (1) Each component  $c_i$  must be essential; that is to say, there exists a realization of the other components  $c_j$ ,  $j \neq i$ , such that  $\phi(1_i, x) = 1$  and  $\phi(0_i, x) = 0$  where  $(\cdot_i, x) = (x_1, \dots, x_{i-1}, \cdot_i, x_{i+1}, \dots, x_n)$ .
- (2) If  $x_i \leq y_i$  for each  $i$ , then  $\phi(x) \leq \phi(y)$ . This condition implies that the state of the system is not degraded by changing a component from a failed condition to a working condition.

From (1) and (2) it immediately follows that  $\phi(1) = 1$  and  $\phi(0) = 0$ .

The state of any component in  $(C, \phi)$  is assumed random with  $P(X_i = 1) = p_i$  and stochastically independent of any other component. The *reliability function*  $h(p)$  is defined to be  $E(\phi(X)) = P(\phi(X) = 1)$ . To further characterize the reliability function  $h(p)$ , the following definitions are needed.

---

<sup>†</sup>Throughout this paper a vector  $(p_1, \dots, p_n)$  is denoted as  $p$  and a scalar function of several variables is designated as  $\phi(x)$  or  $h(p)$ .

- $A \vee B = A + B - AB$ .
- $y < x \rightarrow y_i \leq x_i \quad \forall i$  and  $y_i < x_i$  for some  $i$ .
- *Path Vector of*  $(C, \phi)$ : Vector  $x$  such that  $\phi(x) = 1$ .
- *Cut Vector of*  $(C, \phi)$ : Vector  $x$  such that  $\phi(x) = 0$ .
- *Path Set*:  $\{c_i \mid x_i = 1 \text{ and } \phi(x) = 1\}$ .
- *Cut Set*:  $\{c_i \mid x_i = 0 \text{ and } \phi(x) = 0\}$ .
- *Min Path Vector*: Vector  $x$  such that  $\phi(x) = 1$  and for all  $y < x$ ,  $\phi(y) = 0$ .
- *Min Cut Vector*: Vector  $x$  such that  $\phi(x) = 0$  and for all  $y > x$ ,  $\phi(y) = 1$ .
- *Min Path Set*:  $B = \{c_i \mid x_i = 1 \text{ and } x \text{ is a min path vector}\}$ .
- *Min Path Structure Function*:  $\eta(x) = \prod_{c_i \in B} x_i$  where  $B$  is a min path set.
- *Min Cut Set*:  $A = \{c_i \mid x_i = 0 \text{ and } x \text{ is a min cut vector}\}$ .
- *Min Cut Structure Function*:  $\mu(x) = \bigvee_{c_i \in A} x_i$  where  $A$  is a min cut set.

Birnbaum, Esary, and Saunders [3] show that if  $B_1, \dots, B_r$  comprise the min path sets of  $(C, \phi)$  and  $\eta_i(x)$  comprise the min path structure functions,  $i = 1, \dots, r$ ,

$$h(p) = E \left[ \bigvee_{i=1}^r \eta_i(X) \right] \quad (1)$$

and if  $A_1, A_2, \dots, A_s$  are the min cut sets of  $(C, \phi)$  and  $\mu_j(X)$  make up the min cut structure functions,  $j = 1, 2, \dots, s$ ,

$$h(p) = E \left[ \prod_{j=1}^s \mu_j(X) \right] \quad (2)$$

A method for evaluating (1) and (2) has been proposed by Birnbaum, Esary, and Saunders [3].

A module of a system can be thought of as "a subset of the basic components of the system which are organized into some substructure of their own and which affect the system only through the performance of their substructure. Rephrasing, a module is an assembly of components which can itself be treated as a component of the system."<sup>†</sup> The coherent system  $(A, \chi_A)$  is a *module* of  $(C, \phi)$  if

- $A \subset C$  and  $A$  is not empty.

- $\phi(x) = \psi(\chi_A(x^A), x^{A'})$  for all binary vectors  $x = (x^A, x^{A'})$

where  $A'$  is the complement of  $A$  and  $[c_A \cup A', \psi]$  is a coherent system. In the above definition, all components making up set  $A$  in the coherent structure  $(C, \phi)$  have been replaced by a single component  $c_A$  in the coherent system  $[c_A \cup A', \psi]$ , and the state of  $c_A$  is given by  $\chi_A$ , the structure function of the module  $(A, \chi_A)$ . More generally, the coherent system  $(C, \phi)$  can be decomposed into modules  $(A_i, \chi_{A_i})$ ,  $i = 1, 2, \dots, t$ , such that  $\bigcup_{k=1}^t A_k = C$  and  $A_k \cap A_l = E$ , the empty set, for  $k \neq l$ . Replacing each module  $(A_i, \chi_{A_i})$  by a single component  $M_i$  and denoting the state of  $M_i$  as  $\chi_{A_i}$ , the state of module  $(A_i, \chi_{A_i})$ , a new coherent structure  $[M, \psi]$  is formed where  $M = (M_1, M_2, \dots, M_t)$  and  $\psi = \psi(\chi_{A_1}, \chi_{A_2}, \dots, \chi_{A_t})$ . This reduction is called the *modular decomposition of a coherent structure*.

Since the computation of  $h(p)$  is difficult, a method to approximate  $h(p)$  is desired. Esary and Proschan [5] describe such a procedure. The *Esary-Proschan lower bound procedure* computes a lower bound on  $h(p)$  by enumerating all min cut sets of  $(C, \phi)$ , connecting the components of each min cut set in parallel and joining each of these parallel subsystems in series where the replicated components

<sup>†</sup> Birnbaum and Esary [2].



are replaced by identical but independent operating components. In this paper, the modular decomposition of a coherent structure is utilized together with the Esary-Proschan lower bound procedure to obtain the Lower Bound Modular Decomposition Theorem.

The Lower Bound Modular Decomposition Theorem shows that by decomposing a coherent structure into modules and using the Esary-Proschan lower bound procedure on  $[M, \psi]$  (where the reliability of  $M_1$  is defined to be the *Esary-Proschan lower bound* to  $(A_1, \chi_{A_1})$ ), the lower bound on  $h(p)$  thus found is no worse than applying the Esary-Proschan lower bound procedure to  $(C, \phi)$  directly.

In general, the modular decomposition of a coherent structure is not unique. Hence, the question arises as to which modular decomposition to use. This question is discussed in Section 3.0 by refining the modular decomposition of a coherent structure to include the possibility of decomposing each module further. Finally, in Section 4.0, analogous results are stated for an upper bound on  $h(p)$ .

The following notation is utilized in this paper:

- Reliability of the coherent structure  $(C, \phi) : h(p) = h_C(p) = h_\phi(p)$
- Esary-Proschan lower bound to  $(C, \phi) : h^*(p) = h_C^*(p) = h_\phi^*(p)$

## 2.0 LOWER BOUND MODULAR DECOMPOSITION THEOREM

Let  $\mu_j(x)$ ,  $j = 1, 2, \dots, s$ , be the min cut structure functions of the coherent structure  $(C, \phi)$ . Thus,  $\phi(x) = \prod_{j=1}^s \mu_j(x)$ . The Esary-Proschan lower bound to  $(C, \phi)$  is

$$h_{\phi}^*(p) = \prod_{j=1}^s P(\mu_j(X) = 1) = \prod_{j=1}^s h_{\mu_j}(p) \quad (3)$$

Esary and Proschan [5] show that

$$h(p) = h_{\phi}(p) \geq h_{\phi}^*(p) \quad (4)$$

### Lemma 1:

If  $\chi_1(x), \dots, \chi_t(x)$  are disjoint coherent structure functions and  $\phi(x) = \bigvee_{i=1}^t \chi_i(x)$ , then  $\bigvee_{i=1}^t h_{\chi_i}^*(p) \geq h_{\phi}^*(p)$ .

### Proof:

Let  $\lambda_{i1}(x), \dots, \lambda_{im_i}(x)$  be the min cut structure functions of  $\chi_i(x)$ . Then,  $\chi_i(x) = \prod_{l=1}^{m_i} \lambda_{il}(x)$ . From (3)

$$h_{\chi_i}^*(p) = \prod_{l=1}^{m_i} P(\lambda_{il}(X) = 1) = \prod_{l=1}^{m_i} h_{\lambda_{il}}(p) \quad (5)$$

Let  $\Lambda_{il} = \lambda_{il}(x)$  be independent binary variables,  $l = 1, 2, \dots, m_i$ ,  $i = 1, 2, \dots, t$ . Thus,  $P(\Lambda_{il} = 1) = P(\lambda_{il}(X) = 1) = q_{il}$ . Let  $\chi_i^*(\Lambda) = \prod_{l=1}^{m_i} \Lambda_{il}$  so that

$$h_{\chi_i}^*(p) = h_{\chi_i^*}^*(q) \quad (6)$$

Furthermore, let  $\phi^*(\Lambda) = \bigvee_{i=1}^t \chi_i^*(\Lambda)$ . Since  $\chi_i(x)$ ,  $i = 1, 2, \dots, t$ , are disjoint binary functions,  $\chi_i^*(\Lambda)$  are disjoint binary functions and

$$h_{\phi^*}^*(q) = P\left(\bigvee_{i=1}^t \chi_i^*(\Lambda) = 1\right) = \bigvee_{i=1}^t P(\chi_i^*(\Lambda) = 1) = \bigvee_{i=1}^t h_{\chi_i^*}^*(q) = \bigvee_{i=1}^t h_{\chi_i}^*(p) \quad (7)$$

If  $\mu_j(x)$ ,  $j = 1, 2, \dots, s$ , are the min cut structure functions of  $\phi(x)$ , then by Theorem 4.1 of Birnbaum and Esary [2] for min cut sets, it can be concluded that

$$\mu_j(x) = \bigvee_{i=1}^t \lambda_{i.}(x) \quad (8)$$

for some  $\lambda_{1.}(x), \lambda_{2.}(x), \dots, \lambda_{t.}(x)$ . If we let  $\mu_j^*(\Lambda) = \bigvee_{i=1}^t \lambda_{i.}$ ,

$j = 1, 2, \dots, s$ , where  $\lambda_{i.}(x) = \lambda_{i.}$  in the corresponding expression of (7),

$$h_{\mu_j}^*(p) = h_{\mu_j^*}^*(q) \quad (9)$$

Thus, by (8), (9), and Theorem 4.1 of Birnbaum and Esary [2],  $\mu_1^*(\Lambda)$ ,  $\mu_2^*(\Lambda)$ ,  $\dots, \mu_s^*(\Lambda)$  are the min cut structure functions of  $\phi^*(\Lambda)$ . Then,

$$h_{\phi^*}^*(q) = \prod_{j=1}^s h_{\mu_j^*}^*(q) = \prod_{j=1}^s h_{\mu_j}^*(p) = h_{\phi}^*(p) \quad (10)$$

But, by (7) and (10)

$$\bigvee_{i=1}^t h_{\chi_i}^*(p) = h_{\phi^*}^*(q) \geq h_{\phi^*}^*(q) = h_{\phi}^*(p) \quad (11)$$

which proves the lemma. //

Let  $h_x = [h_{x_1}(p), \dots, h_{x_t}(p)]$  and  $h_x^* = [h_{x_1}^*(p), \dots, h_{x_t}^*(p)]$ .

**Theorem 2:** *Lower Bound Modular Decomposition Theorem*

If  $x_1(x), x_2(x), \dots, x_t(x)$  are disjoint coherent structure functions, then

$$\begin{aligned}
 h_\phi(p) = h_\psi[h_x] &\geq \text{Max} \{h_\psi^*[h_x], h_\psi[h_x^*]\} \\
 &\geq \text{Min} \{h_\psi^*[h_x], h_\psi[h_x^*]\} \\
 &\geq h_\psi^*[h_x^*] \\
 &\geq h_\phi^*(p)
 \end{aligned} \tag{12}$$

**Proof:**

Recall that  $\psi = \psi(x_{A_1}, x_{A_2}, \dots, x_{A_t})$  so that  $h_\phi(p) = h_\psi[h_x]$  by (6.21) of Birnbaum and Esary [2]. Since  $h_{x_i}(p) \geq h_{x_i}^*(p)$  by (4) and  $h_\psi[u_1, u_2, \dots, u_t]$  is monotone in  $u_i$  for each  $i$ ,  $h_\phi(p) \geq h_\psi[h_x^*]$  and  $h_\psi^*[h_x] \geq h_\psi^*[h_x^*]$ . By (4),  $h_\phi(p) \geq h_\psi^*[h_x]$  and  $h_\psi[h_x^*] \geq h_\psi^*[h_x^*]$ . If we can show  $h_\psi^*[h_x^*] \geq h_\phi^*(p)$ , we have established the theorem.

Let  $\phi_j(x) = \xi_j(x_1(x), \dots, x_t(x))$ ,  $j = 1, 2, \dots, s$ , be the min cut structure functions of the coherent structure  $[M, \psi]$  where  $\phi(x) = \psi[x_1(x), \dots, x_t(x)]$ . Let  $\mu_{j\ell}(x)$ ,  $\ell=1, 2, \dots, K_j$ , be the min cut structure functions of  $\phi_j(x)$ . It is easy to see that  $\mu_{j\ell}(x)$  are the min cut structure functions of  $\phi(x)$ ,  $\ell = 1, 2, \dots, K_j$ ,  $j = 1, 2, \dots, s$ . By Lemma 1,

$$h_{\xi_j}[h_x^*] \geq h_{\phi_j}^*(p) = \prod_{\ell=1}^{K_j} h_{\mu_{j\ell}}(p) \tag{13}$$

Hence,

$$h_{\psi}^*[h_{\chi}^*] = \prod_{j=1}^s h_{\xi_j}^*[h_{\chi}^*] \geq \prod_{j=1}^s h_{\phi_j}^*(p) = \prod_{j=1}^s \prod_{l=1}^{K_j} h_{\mu_{jl}}^*(p) = h_{\phi}^*(p) \quad (14)$$

This proves the theorem. //

Example 1:

Let the coherent structure  $(C, \phi)$  be given as in Figure 1.

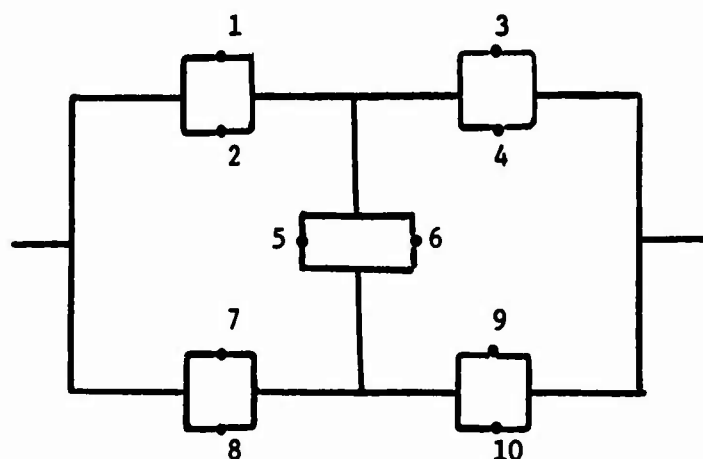


FIGURE 1

Let  $(A_i, \chi_i)$  be the modular decomposition of  $(C, \phi)$  where  $A_i = (C_{2i-1}, C_{2i})$ ,  $i=1,2,3,4,5$ . Then,

$$h_{\phi}(p) = h_{\psi}[h_{\chi}^*] \geq h_{\psi}^*[h_{\chi}] = h_{\psi}^*[h_{\chi}^*] = h_{\phi}^*(p) \quad (15)$$

The coherent structure which generates  $h_{\psi}^*[h_{\chi}]$ ,  $h_{\psi}^*[h_{\chi}^*]$  and  $h_{\phi}^*(p)$  is given in Figure 2.

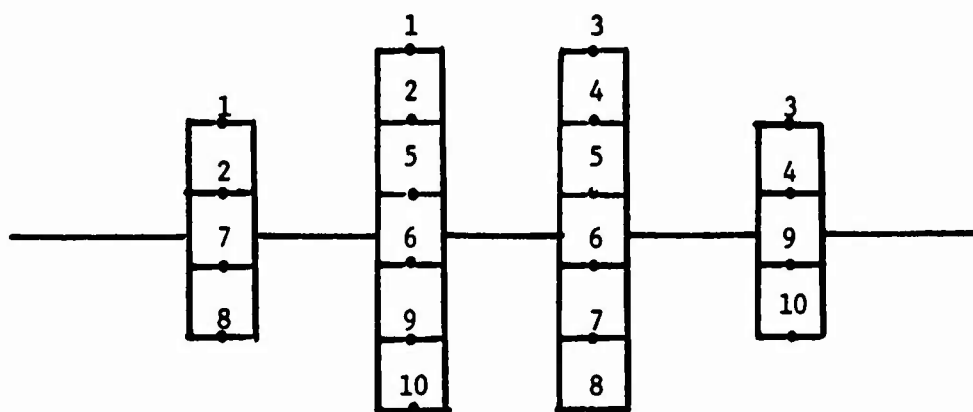


FIGURE 2

The method suggested by the Lower Bound Modular Decomposition Theorem has two inherent advantages over the Esary-Proschan procedure:

- (1) It is a more accurate estimate of  $h_{\phi}(p)$ .
- (2) It requires the enumeration of all min cut sets over a set of coherent structures with less components. Since the work required to enumerate all min cut sets of a coherent structure increases exponentially with the number of components in the structure, this enumeration can be carried out more efficiently.

To illustrate the accuracy of the approximations to system reliability obtained by utilizing the Lower Bound Modular Decomposition Theorem as opposed to the Esary-Proschan procedure applied directly to the coherent structure, consider the following example.

Example 2:

Figure 3 illustrates the coherent structure under consideration.

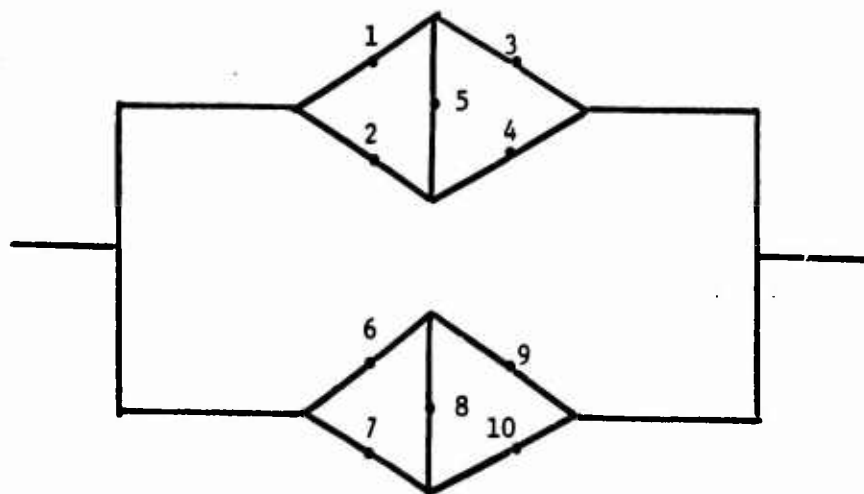


FIGURE 3

Define  $(A_i, \chi_i)$ ,  $i=1,2$ , to be the modular decomposition of  $(C, \phi)$  where the modular sets are  $A_1 = \{1,2,3,4,5\}$  and  $A_2 = \{6,7,8,9,10\}$ . Applying the Lower Bound Modular Decomposition Theorem, we find that,

$$\begin{aligned}
 h_{\phi}(p) &= h_{\psi}^*[h_1(p), h_2(p)] \\
 &\geq h_{\psi}[h_1^*(p), h_2^*(p)] \\
 &= h_{\psi}^*[h_1^*(p), h_2^*(p)] \\
 &\geq h_{\phi}^*(p)
 \end{aligned}
 \tag{15}$$

Assuming each component to have the same reliability, we obtain the following table (Figure 4).

| Component Reliability | $h_{\phi}(p)$          | $h_{\psi}^*[h_1^*(p), h_2^*(p)]$ | $h_{\phi}^*(p)$          |
|-----------------------|------------------------|----------------------------------|--------------------------|
| .99                   | .99999996              | .99999996                        | .99999996                |
| .95                   | .99997275              | .99997251                        | .99997243                |
| .90                   | .99953689              | .99952217                        | .99951609                |
| .75                   | .98077010              | .97799376                        | .97584785                |
| .50                   | .75                    | .67585658                        | .56262773                |
| .25                   | .25811386              | .12385429                        | .011416517               |
| .10                   | .042576889             | .00529541                        | .533 x 10 <sup>-6</sup>  |
| .01                   | .40 x 10 <sup>-3</sup> | .698 x 10 <sup>-6</sup>          | .941 x 10 <sup>-21</sup> |

FIGURE 4

### 3.0 EXTENSIONS

Since the modular decomposition of a coherent structure is not necessarily unique, the question arises as to which modular decomposition to utilize. Insight into this question is given by the results of this section.

Let  $w_{ij}(x)$ ,  $j=1,2,\dots,s_i$ ,  $i=1,2,\dots,t$  be disjoint coherent structure functions and  $\sigma_i(w)$ ,  $i=1,2,\dots,t$  be disjoint coherent structure functions such that  $\chi_i(x) = \sigma_i(w_{i1}(x), w_{i2}(x), \dots, w_{is_i}(x))$ . Let  $\theta(w)$  be the coherent structure function defined by  $\theta(w) = \psi(\sigma_1(w), \dots, \sigma_t(w))$ . Then

$$\begin{aligned}\theta(x) &= \psi(\chi_1(x), \dots, \chi_t(x)) \\ &= \psi\left[\sigma_1(w_{11}(x), \dots, w_{1s_1}(x)), \dots, \sigma_t(w_{t1}(x), \dots, w_{ts_t}(x))\right] \\ &= \theta(w_{11}(x), \dots, w_{ts_t}(x))\end{aligned}\quad (16)$$

Thus,  $\chi_1(x), \dots, \chi_t(x)$  define the coherent structure functions of a modular decomposition of  $(C, \phi)$  while  $w_{i1}(x), \dots, w_{is_i}(x)$  define the coherent structure functions of a modular decomposition of the coherent structure defined by  $\chi_i(x)$ . Let  $h_{w_i} = [h_{w_{i1}}(p), \dots, h_{w_{is_i}}(p)]$  and  $h_{w_i}^* = [h_{w_{i1}}^*(p), \dots, h_{w_{is_i}}^*(p)]$ ,  $i=1,2,\dots,t$ .

#### Theorem 3:

$$\begin{aligned}(a) \quad h_{\psi}^*[h_{\chi}] &\geq h_{\theta}^*[h_{w_1}, h_{w_2}, \dots, h_{w_t}] \\ (b) \quad h_{\psi}[h_{\chi}^*] &\leq h_{\theta}[h_{w_1}^*, h_{w_2}^*, \dots, h_{w_t}^*]\end{aligned}\quad (17)$$



Proof:

(a)  $w_{ij}(X)$ ,  $j=1,2, \dots, s_i$ ,  $i=1,2, \dots, r$ , are independent binary random variables. Let  $q_{ij} = P[w_{ij}(X) = 1]$ ,  $j=1,2, \dots, s_i$ ,  $i=1,2, \dots, r$ . By the Lower Bound Modular Decomposition Theorem,

$$h_{\psi}^*[h_{\sigma_1}(q), \dots, h_{\sigma_r}(q)] \geq h_{\theta}^*(q) \quad (18)$$

Now,

$$\begin{aligned} h_{\psi}^*[h_X] &= h_{\psi}^*[h_{\sigma_1}(q), \dots, h_{\sigma_r}(q)] \\ &\geq h_{\theta}^*(q) = h_{\theta}^*[h_{w_1}, \dots, h_{w_t}] \end{aligned} \quad (19)$$

(b) By the Lower Bound Modular Decomposition Theorem

$$h_{\chi_i}^*(p) \leq h_{\sigma_i}[h_{w_i}^*], \quad i=1,2, \dots, t \quad (20)$$

Since  $h_{\psi}$  is nondecreasing,

$$\begin{aligned} h_{\psi}[h_X^*] &\leq h_{\psi}[h_{\sigma_1}(h_{w_1}^*), \dots, h_{\sigma_t}(h_{w_t}^*)] \\ &= h_{\theta}[h_{w_1}^*, \dots, h_{w_t}^*] \quad // \end{aligned} \quad (21)$$

The modular decomposition of  $(C, \phi)$  defined by the coherent structure functions  $w_{ij}(X)$  is a refinement of the modular decomposition of  $(C, \phi)$  defined by the coherent structure functions  $\chi_i(x)$ . Let  $\phi(x) = \psi_k[\chi_{1k}(x), \dots, \chi_{r_k k}(x)]$ ,  $k=0, \dots, k^*$ , be a series of increasingly refined decompositions where  $r_0 = 1$ ,  $\chi_{1k}(x) = \phi(x)$ ,  $\psi_0[\phi(x)] = \phi(x)$ ,  $r_{k^*} = n$ , the number of components in the coherent structure,  $\chi_{1k^*} = x_1$ ,  $i=1,2, \dots, n$ ,  $\psi_{k^*}[\chi_{1k^*}, \dots, \chi_{nk^*}] = \phi(x)$  and  $\psi_{k^*}^*[\chi_{1k^*}, \dots, \chi_{nk^*}] = \phi^*(x)$ .

Then, from Theorem 3,

$$h_{\psi}^*[h_{\chi}]^{(k)} = h_{\psi_k}^*[h_{\chi_{1k}}(p), \dots, h_{\chi_{r_k k}}(p)] \text{ is nonincreasing in } k, k=0,1,2, \dots, k^*$$

$$h_{\psi}^*[h_{\chi}^*]^{(k)} = h_{\psi_k}^*[h_{\chi_{1k}}^*(p), \dots, h_{\chi_{r_k k}}^*(p)] \text{ is nondecreasing in } k, k=0,1,2, \dots, k^*$$

where

$$\begin{aligned} h_{\psi}^*[h_{\chi}]^{(0)} &= h_{\phi}(p) \\ h_{\psi}^*[h_{\chi}]^{(k^*)} &= h_{\phi}^*(p) \\ h_{\psi}^*[h_{\chi}^*]^{(0)} &= h_{\phi}^*(p) \\ h_{\psi}^*[h_{\chi}^*]^{(k^*)} &= h_{\phi}(p) \end{aligned} \tag{22}$$

Let,

$$h_{\psi}^*[h_{\chi}^*]^{(k)} = h_{\psi_k}^*[h_{\chi_{1k}}^*(p), \dots, h_{\chi_{r_k k}}^*(p)] \tag{23}$$

Then, by the Lower Bound Modular Decomposition Theorem

$$h_{\psi}^*[h_{\chi}^*]^{(k)} \leq \text{Min} [h_{\psi}^*(h_{\chi})^{(k)}, h_{\psi}(h_{\chi}^*)^{(k)}], k=0,1,2, \dots, k^* \tag{24}$$

These results can be qualitatively depicted as follows (Figure 5).

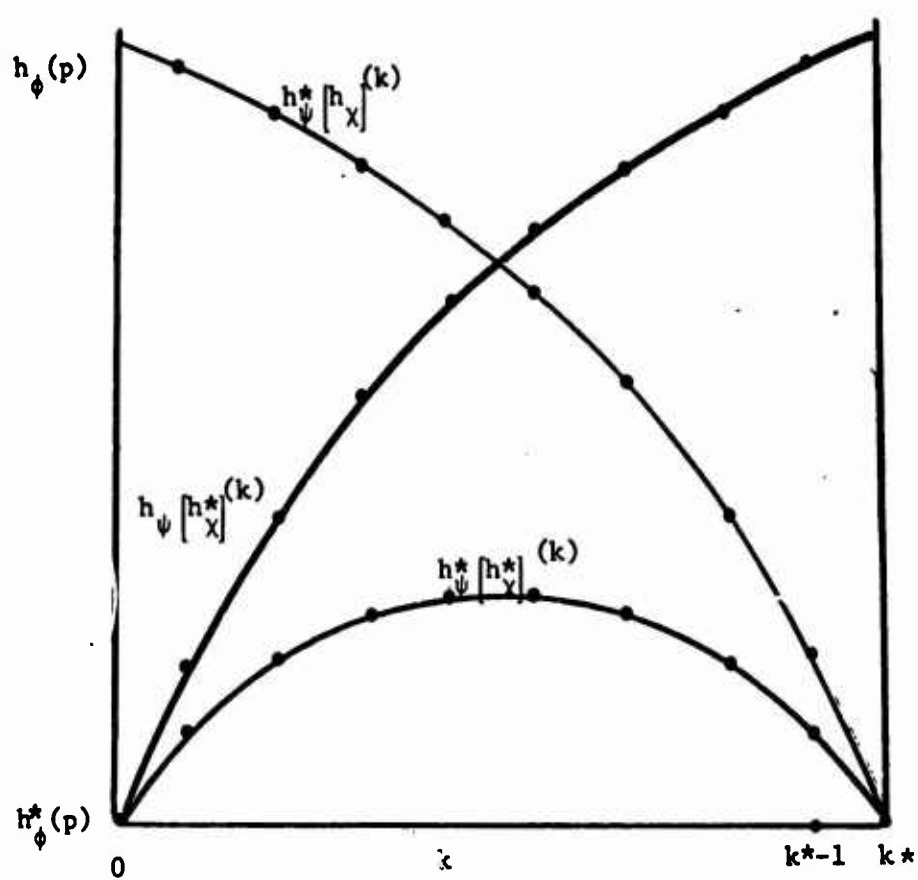


FIGURE 5

#### 4.0 UPPER BOUND MODULAR DECOMPOSITION THEOREM

Let  $\eta_j(x)$ ,  $j=1,2,\dots,r$ , be the min path structure functions of  $(C, \phi)$  where  $\phi(x) = \bigvee_{j=1}^r \eta_j(x)$ . The Esary-Proschan upper bound to  $(C, \phi)$  is

$$h_{\phi}^{**}(p) = \bigvee_{j=1}^r P(\eta_j(X) = 1) = \bigvee_{j=1}^r h_{\eta_j}(p) \quad (25)$$

where

$$h_{\phi}^{**}(p) \geq h(p) \quad (26)$$

Results similar to those derived previously in this paper can be shown. These results are stated without proof since the proofs are analogous to those given previously.

##### Lemma 5:

If  $\chi_1(x), \chi_2(x), \dots, \chi_t(x)$  are disjoint coherent structure functions and  $\phi(x) = \prod_{i=1}^t \chi_i(x)$ , then  $\prod_{i=1}^t h_{\chi_i}^{**}(p) = h_{\phi}^{**}(p)$ .

##### Theorem 6: Upper Bound Modular Decomposition Theorem

If  $\chi_1(x), \dots, \chi_t(x)$  are disjoint coherent structure functions, then,

$$\begin{aligned} h_{\phi}(p) = h_{\psi[h_{\chi}]} &\leq \text{Min}\{h_{\psi}^{**}[h_{\chi}], h_{\psi}[h_{\chi}^{**}]\} \\ &\leq \text{Max}\{h_{\psi}^{**}[h_{\chi}], h_{\psi}[h_{\chi}^{**}]\} \\ &\leq h_{\psi}^{**}[h_{\chi}^{**}] \\ &\leq h_{\phi}^{**}(p) \end{aligned} \quad (27)$$

where  $h_{\chi} = [h_{\chi_1}(p), \dots, h_{\chi_t}(p)]$  and  $h_{\chi}^{**} = [h_{\chi_1}^{**}(p), \dots, h_{\chi_t}^{**}(p)]$ .

Theorem 7:

$$\begin{aligned} (a) \quad h_{\psi}^{**} [h_{\chi}] &\leq h_{\theta}^{**} [h_{w_1}, \dots, h_{w_t}] \\ (b) \quad h_{\psi}^{**} [h_{\chi}] &\geq h_{\theta}^{**} [h_{w_1}^{**}, \dots, h_{w_t}^{**}] \end{aligned} \quad (28)$$

where the coherent structure functions  $w_{ij}(x)$  are defined as in Section 3.0,

$$h_{w_i} = [h_{w_{i1}}(p), \dots, h_{w_{is_i}}(p)] \quad , \quad \text{and} \quad h_{w_i}^{**} = [h_{w_{i1}}^{**}(p), \dots, h_{w_{is_i}}^{**}(p)] \quad .$$

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